Intrinsic Brain Surface Conformal Mapping using a Variational Method

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Abstract. We propose a new variational method which can find a unique conformal mapping between any two genus zero manifolds by minimizing the harmonic energy of the map and apply it to the cortical surface matching problem. We use a mesh structure to represent the brain surface. Empirical tests on MRI data show that the mappings preserve angular relationships, are stable in MRIs acquired at different times, and are robust to differences in data triangulation, and resolution.

1 Introduction

Recent developments in brain imaging have accelerated the collection and databasing of brain maps. Nonetheless, computational problems arise when integrating and comparing brain data. One way to analyze and compare brain data is to map them into a canonical space while retaining geometric information on the original structures as far as possible [1–4].

Conformal surface parameterization has been studied intensively. Most works in conformal parameterization deal with surface patches homeomorphic to topological disks. For surfaces with arbitrary topologies, Gu and Yau [5] introduce a general method for global conformal parameterization, which is based on the structure of the cohomology group of holomorphic one-forms. They generalize the method for surfaces with boundaries in [6].

For genus zero surfaces, there are four basic approaches to achieve the conformal parameterization. *Harmonic energy minimization.* Eck et al. [7] introduce the discrete harmonic map, which approximates the continuous harmonic maps [8] by minimizing a *metric dispersion* criterion. Desbrun et al. [9, 10] compute the discrete Dirichlet energy and apply conformal parameterization to interactive geometry remeshing. Gu and Yau in [5] introduce a non-linear optimization method to compute global conformal parameterizations for genus zero surfaces. The optimization is carried out in the tangential spaces of the sphere. *Cauchy-Riemann equation approximation.* Levy et al. [11] compute a quasi-conformal parameterization of topological disks by approximating the Cauchy-Riemann equation using the least squares method. *Laplacian operator linearization.* Haker et al. [2] use a method to compute a global conformal mapping from a genus zero surface to a sphere by representing the Laplace-Beltrami operator as a linear system. *Circle packing.* Classical analytic functions can be approximated using circle packings. Hurdal et al. [1] use it to create flat maps of the brain.

Bakircioglu et al. use spherical harmonics to compute a flow on the sphere in [12] in order to match curves on the brain. Thompson and Toga use a similar approach in [13]. This flow field can be thought of as the variational minimizer of the integral over the sphere of Lu, with L some power of the Laplacian, and u the deformation. This is very similar to the spherical harmonic map used in this paper.

Any genus zero surface can be mapped conformally onto the sphere and any local portion thereof onto a disk. This mapping, a conformal equivalence, is one-to-one, onto, and angle-preserving. Moreover, the elements of the first fundamental form remain unchanged, except for a scaling factor (the so-called *Conformal Factor*). In this paper, we propose a new genus zero surface conformal mapping algorithm [5] and demonstrate its use in computing conformal mappings between brain surfaces.

Suppose K is a simplicial complex, and $f : |K| \to R^3$, which embeds $|K|$ in R^3; then (K, f) is called a mesh. Our algorithm for computing conformal mappings is based on the fact that for genus zero surfaces S_1, S_2, $f : S_1 \to S_2$ is conformal if and only if f is harmonic. All conformal mappings between S_1, S_2 form a group, the so-called Möbius group. Our method is as follows: we first find a homeomorphism h between M_1 and M_2, then deform h such that h minimizes the harmonic energy. To ensure the convergence of the algorithm, constraints are added; this also ensures that there is a unique conformal map.

This paper is organized as follows. In Section 2, we give various definitions which are used in our variational method. In Section 3, we detail our conformal spherical mapping algorithms. In Section 4, we propose a method to optimize the conformal parameterization by landmarks. Experimental results on conformal mapping for brain surfaces are reported in Section 5. We compare our algorithm with other conformal mapping approaches used in neuroimaging and conclude the paper in Section 6.

2 Variational Method

We assume the brain surfaces are represented as triangulated meshes, and denoted as K, the vertices are u, v, the edges are $[u, v]$.

Definition 1. *Suppose $\boldsymbol{f} : K \to R^3$, a function defined on the vertices of K, the string energy is defined as:*

$$E(\boldsymbol{f}) = \sum_{[u,v]\in K} k_{u,v} ||\boldsymbol{f}(u) - \boldsymbol{f}(v)||^2 \tag{1}$$

By changing the string constants $k_{u,v}$ in the energy formula, we can define different string energies. $|| \cdot ||^2$ is the traditional Euclidean norm.

Definition 2. *If string constants $k(u, v) \equiv 1$, the string energy is known as the Tutte energy.*

Definition 3. *Suppose angles* α, β *are the two angles against edge* $[u, v]$

$$k_{u,v} = \frac{1}{2}(\cot \alpha + \cot \beta), \tag{2}$$

the string energy obtained is called the harmonic energy.

Definition 4. *The piecewise Laplacian is a function defined on the vertices of* K, $\Delta : K \rightarrow R^3$

$$\Delta \boldsymbol{f}(u) = \sum_{[u,v] \in K} k_{u,v}(\boldsymbol{f}(v) - \boldsymbol{f}(u)) \tag{3}$$

If $\boldsymbol{f}$ minimizes the string energy, then $\boldsymbol{f}$ is said to be *harmonic*. A map $\boldsymbol{f} : M_1 \rightarrow M_2$ is harmonic, if and only if it only has a normal component, and the tangential component is zero.

$$\Delta \boldsymbol{f} = \Delta \boldsymbol{f}^{\perp} \tag{4}$$

Suppose $\boldsymbol{f} : M_1 \rightarrow M_2$, and denote the image of each vertex $v \in K_1$ as $\boldsymbol{f}(v)$. The normal on M_2 at $\boldsymbol{f}(v)$ is $\boldsymbol{n}(\boldsymbol{f}(v))$.The normal component of the Laplacian is defined as the following:

Definition 5. *The normal component*

$$(\Delta \boldsymbol{f}(v))^{\perp} =< \Delta \boldsymbol{f}(v), \boldsymbol{n}(\boldsymbol{f}(v)) > \boldsymbol{n}(\boldsymbol{f}(v)), \tag{5}$$

where $<,>$ *is the inner product in* R^3.

Definition 6. *The absolute derivative is defined as*

$$D\boldsymbol{f}(v) = \ \Delta \boldsymbol{f}(v) - (\Delta \boldsymbol{f}(v))^{\perp} \tag{6}$$

Suppose we would like to compute a mapping $\boldsymbol{f} : M_1 \rightarrow M_2$ such that $\boldsymbol{f}$ minimizes a string energy $E(\boldsymbol{f})$. This can be solved easily by the steepest descent algorithm:

$$\frac{d\boldsymbol{f}(t)}{dt} = -D\boldsymbol{f}(t) \tag{7}$$

or equivalently $\delta \boldsymbol{f} = -D\boldsymbol{f} \times \delta t$, the steady state solution is the desired harmonic map.

3 Conformal Spherical Mapping

Suppose M_2 is S^2, then a conformal mapping $\boldsymbol{f} : M_1 \rightarrow S^2$ can be constructed by using the steepest descent method. The major difficulty is that the solution is not unique but forms a Möbius group.

Definition 7. *Mapping* $f : C \rightarrow C$ *is a Möbius transformation if and only if*

$$f(z) = \frac{az + b}{cz + d}, \ \ a, b, c, d \in C, ad - bc \neq 0 \tag{8}$$

All Möbius transformations form the Möbius transformation group. In order to determine a unique solution we can add different constraints. In practice we use the following two constraints: zero mass-center constraint and a landmark constraint.

Definition 8. *Mapping* $\boldsymbol{f} : M_1 \to M_2$ *satisfies the zero mass-center condition if and only if*

$$\int_{M_2} \boldsymbol{f} d\sigma_{M_1} = 0, \tag{9}$$

where σ_{M_1} *is the area element on* M_1*.*

All conformal maps from M_1 to S^2 satisfying the zero mass-center constraint are unique up to a Euclidean rotation group (which is 3 dimensional). We use the Gauss map as the initial condition.

Definition 9. *A Gauss map* $N : M_1 \to S^2$ *is defined as*

$$N(v) = \boldsymbol{n}(v), v \in M_1, \tag{10}$$

$\boldsymbol{n}(v)$ *is the normal at* v*.*

Algorithm 1 *Spherical Tutte Mapping*

Input (mesh M*,step length* δt*, energy difference threshold* δE*), output(*$\boldsymbol{t} : M \to S^2$*) where* $\boldsymbol{t}$ *minimizes the Tutte energy.*

1. *Compute Gauss map* $N : M \to S^2$*. Let* $\boldsymbol{t} = N$*, compute Tutte energy* E_0*.*
2. *For each vertex* $v \in M$*, compute Absolute derivative* $D\boldsymbol{t}$*.*
3. *Update* $\boldsymbol{t}(v)$ *by* $\delta \boldsymbol{t}(v) = -D\boldsymbol{t}(v)\delta t$*.*
4. *Compute Tutte energy* E*.*
5. *If* $E - E_0 < \delta E$*, return* $\boldsymbol{t}$*. Otherwise, assign* E *to* E_0 *and repeat steps* 2 *through to* 5*.*

Because the Tutte algorithm converges rapidly and is stable. We use it as the initial condition for the conformal mapping.

Algorithm 2 *Spherical Conformal Mapping*

Input (mesh M*,step length* δt*, energy difference threshold* δE*), output(*$\boldsymbol{h} : M \to S^2$*). Here* $\boldsymbol{h}$ *minimizes the harmonic energy and satisfies the zero mass-center constraint.*

1. *Compute Tutte embedding* $\boldsymbol{t}$*. Let* $\boldsymbol{h} = \boldsymbol{t}$*, compute Tutte energy* E_0*.*
2. *For each vertex* $v \in M$*, compute the absolute derivative* $D\boldsymbol{h}$*.*
3. *Update* $\boldsymbol{h}(v)$ *by* $\delta \boldsymbol{h}(v) = -D\boldsymbol{h}(v)\delta t$*.*
4. *Compute Möbius transformation* $\boldsymbol{\varphi}_0 : S^2 \to S^2$*, such that*

$$\Gamma(\boldsymbol{\varphi}) = \int_{S^2} \boldsymbol{\varphi} \circ \boldsymbol{h} d\sigma_{M_1}, \boldsymbol{\varphi} \in Mobius(CP^1) \tag{11}$$

$$\boldsymbol{\varphi}_0 = \min_{\boldsymbol{\varphi}} ||\Gamma(\boldsymbol{\varphi})||^2 \tag{12}$$

where σ_{M_1} is the area element on M_1. $\Gamma(\boldsymbol{\varphi})$ is the mass center, $\boldsymbol{\varphi}$ minimizes the norm of mass center.
5. *compute the conformal energy E.*
6. *If $E - E_0 < \delta E$, return $\boldsymbol{t}$. Otherwise, assign E to E_0 and repeat step 2 through to step 6.*

Step 4 is non-linear and expensive to compute. In practice we use the following procedure to replace it:

1. Compute the mass center $\boldsymbol{c} = \int_{S^2} \boldsymbol{h} d\sigma_{M_1}$;
2. For all $v \in M$, $\boldsymbol{h}(v) = \boldsymbol{h}(v) - \boldsymbol{c}$;
3. For all $v \in M$, $\boldsymbol{h}(v) = \frac{\boldsymbol{h}(v)}{||\boldsymbol{h}(v)||}$.

This approximation method is good enough for our purpose. By choosing the step length carefully, the energy can be decreased monotonically at each iteration.

4 Optimize the Conformal Parameterization by Landmarks

In order to compare two brain surfaces, it is desirable to adjust the conformal parameterization and match the geometric features on the brains as well as possible. We define an energy to measure the quality of the parameterization. Suppose two brain surfaces S_1, S_2 are given, conformal parameterizations are denoted as $f_1 : S^2 \to S_1$ and $f_2 : S^2 \to S_2$, the *matching energy* is defined as

$$E(f_1, f_2) = \int_{S^2} ||f_1(u,v) - f_2(u,v)||^2 du dv \tag{13}$$

We can composite a Möbius transformation τ with f_2, such that

$$E(f_1, f_2 \circ \tau) = \min_{\zeta \in \Omega} E(f_1, f_2 \circ \zeta), \tag{14}$$

where Ω is the group of Möbius transformations. We use landmarks to obtain the optimal Möbius transformation. Landmarks are commonly used in brain mapping. We manually label the landmarks on the brain as a set of sulcal curves [3], as shown in Figure 1(i) and (k). First we conformally map two brains to the sphere, then we pursue a best Möbius transformation to minimize the Euclidean distance between the corresponding landmarks on the spheres. Suppose the landmarks are represented as discrete point sets, and denoted as $\{p_i \in S_0\}$ and $\{q_i \in S_1\}$, p_i matches q_i, $i = 1, 2, \dots, n$. The landmark mismatch functional for $u \in \Omega$ is defined as

$$E(u) = \sum_{i=1}^{n} ||p_i - u(q_i)||^2, u \in \Omega, p_i, q_i \in S^2 \tag{15}$$

In general, the above variational problem is a nonlinear one. In order to simplify it, we convert it to a least squares problem by restricting τ in a subgroup of Ω, such that the north pole is the fixed point under this subgroup.

5 Experimental Results

The 3D brain meshes are reconstructed from 3D 256x256x124 T1 weighted SPGR (spoiled gradient) MRI images, by using an active surface algorithm that deforms a triangulated mesh onto the brain surface [4]. Figure 1(a) and (b) show the same brain scanned at different times [3]. Because of the inaccuracy introduced by scanner noise in the input data, as well as slight biological changes over time, the geometric information is not exactly the same. Figure 1(a) and (b) reveal minor differences. The conformal mapping results are shown in Figure 1(c) and (d). From this example, we can see that although the brain meshes are slightly different, the mapping results look quite similar.

Figure 1(e) and (f) show the mapping is conformal by texture mapping a checker board to both the brain surface mesh and a spherical mesh. Each black or white square in the texture is mapped to sphere by stereographic projection, and pulled back to the brain. Note that the right angles are preserved both on the sphere and the brain. In order to measure the conformality, we map the iso-polar angle curves and iso-azimuthal angle curves from the sphere to the brain by the inverse conformal mapping, and measure the intersection angles on the brain. The distribution of the angles of a subject(A) are illustrated in Figure 1 (g) and (h). The angles are concentrated about the right angle.

Figure 1(i-l) show the landmarks, and the result of the optimization by a Möbius transformation. We also computed matching energy following Equation 13. We did our testing among three subjects. Their information are shown in Table 1. We took subject A as the target brain. For each new subject model, we found a Möbius transformation which minimized the landmark mismatch energy on the maximum intersection subsets of it and A. As shown in Table 1, the matching energies were reduced after the Möbius transformation.

The method described in this work is very general. We tested the algorithm on other genus zero surfaces, including the hand and foot surface. The result is illustrated in Figure 1(m-p).

6 Conclusion and Future work

In this paper, we propose a general method which finds a unique conformal mapping between genus zero manifolds. Specifically, we demonstrate its feasibility

Subject	Vertex #	Face #	Before	After
A	65,538	131,072	-	-
B	65,538	131,072	604.134	506.665
C	65,538	131,072	414.803	365.325

Table 1: Matching energy for three subjects. Subject A was used as the target brain. For subjects B and C, we found Möbius transformations that minimized the landmark mismatch functions, respectively.

for brain surface conformal mapping research. Our method only depends on the surface geometry and not on the mesh structure (i.e. gridding) and resolution. Our algorithm is very fast and stable in reaching a solution.

Other studies of conformal mappings between brain surfaces are reported in [1, 2]. Compared with [1], our method really preserves angles and establishes a good mapping between brains and a canonical space. Compared with the work in [2], our method is more accurate, with no regions of large area distortion. It is also more stable and can be readily extended to compute maps between two general manifolds.

There are numerous applications of these mapping algorithms, such as providing a canonical space for automated feature identification, brain to brain registration, brain structure segmentation, brain surface denoising, and convenient surface visualization, among others. We are trying to further study these applications using our variational method based brain conformal mapping technique.

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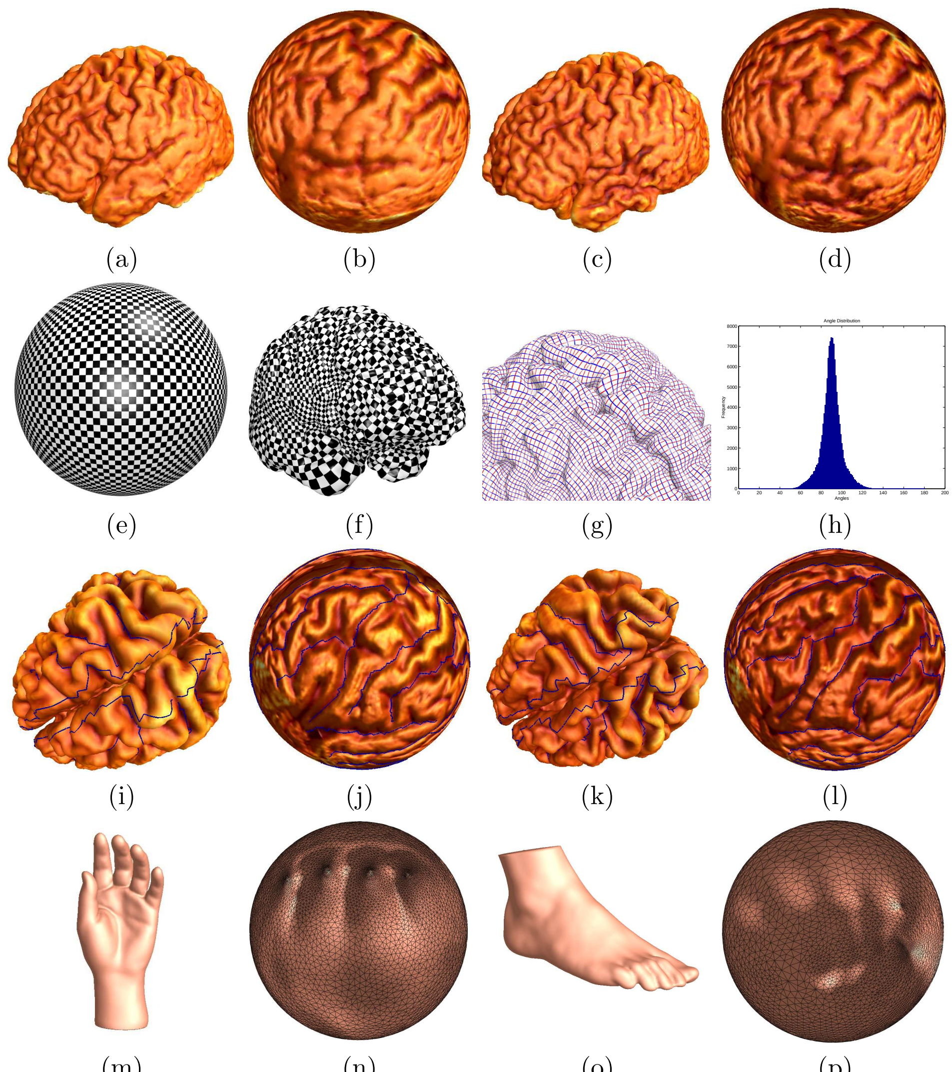

Fig. 1: **Row 1:** Reconstructed brain meshes and their spherical harmonic mappings. (a) and (b) are the reconstructed surfaces for the same brain scanned at different times. Due to scanner noise and inaccuracy in the reconstruction algorithm, there are visible geometric differences. (c) and (d) are the spherical conformal mappings of (a) and (b) respectively; the normal information is preserved. By the shading information, the correspondence is illustrated. **Row 2:** Conformal texture mapping and Conformality measurement. The conformality is visualized by texture mapping of a checkerboard image((e) and (f)). All the right angles on the texture are preserved on the brain surface. The curves of iso-polar angle and iso-azimuthal angle are mapped to the brain, and the intersection angles are measured on the brain(g). The histogram is illustrated in (h) **Row 3:** Möbius transformation to minimize the deviations between landmarks((i), (j), (k) and (l)). The blue curves are the landmarks. The correspondence between curves has been preassigned. The desired Möbius transformation is obtained to minimize the matching error on the sphere. **Row 4:** Spherical conformal mapping of general genus zero surfaces((m),(n), (o) and (p)). Extruding parts (such as fingers and toes) are mapped to denser regions on the sphere.