Figure 17.12: Maps of the human cerebral cortex: flat maps, spherical maps, and Tensor Maps. Extreme variations in cortical anatomy (3D Models; top left) present challenges in brain mapping because of the need to compare cortically-derived brain maps from many subjects. Comparisons of cortical geometry can be based on the warped mapping of one subject's cortex onto another (top right; [25, 26]). These warps can also transfer functional maps from one subject to another, or onto a common template for comparison. Current approaches for deforming one cortex into the shape of another typically simplify the problem by first representing cortical features on a 2D plane, sphere, or ellipsoid, where the matching procedure (i.e., finding $\mathbf{u}^*(r)$ above) is subsequently performed. In one approach [25, 26], active surface extraction of the cortex provides a continuous inverse mapping from the cortex of each subject to the spherical template used to extract it. These inverse maps are applied to connected networks of curved sulci in each subject. This transforms the problem into one of computing an angular flow vector field $\mathbf{u}^*(r)$, in spherical coordinates, which drives the network elements into register on the sphere (middle panel; [20]). The full mapping (top right) can be recovered in 3D space as a displacement vector field which drives cortical points and regions in one brain into register with their counterparts in the other brain. (For a color version of this Figure see Plate 34 in the color section of this book.)
Figure 17.13: Gyral pattern matching. Gyral patterns can be matched in a group of subjects to create average cortical surfaces. (a) shows a cortical flat map for the left hemisphere of one subject, with the average cortical pattern for the group overlaid (colored lines). (b) shows the result of warping the individual’s sulcal pattern into the average configuration for the group, using the covariant field equations (see main text). The individual cortex (a) is reconfigured (b) to match the average set of cortical curves. The 3D cortical regions that map to these average locations are then recovered in each individual subject, as follows. A color code (c) representing 3D cortical point locations (e) in this subject is convected along with the flow that drives the sulcal pattern into the average configuration for the group (d). Once this is done in all subjects, points on each individual’s cortex are recovered (f) that have the same relative location to the primary folding pattern in all subjects. Averaging of these corresponding points results in a well-resolved average cortex. These transformation fields are stored and used to measure regional variability. (For a color version of this Figure see Plate 35 in the color section of this book.)
17.7.3 Covariant field equations

Since the cortex is not a developable surface, it cannot be given a parameterization whose metric tensor is uniform. We therefore developed an approach to take into account the intrinsic curvature of the solution domain when computing flow vector fields in the cortical parameter space. Similar approaches are common in fluid dynamics or general relativity applications, as they make sure that the mapping of one mesh surface onto another is parameterization-invariant. In the covariant tensor approach [33,34], correction terms (Christoffel symbols, $\Gamma^i_{jk}$) make the necessary adjustments for fluctuations in the metric tensor of the mapping procedure. In the partial differential equations [Eq. (17.23)], we replace $L$ by the covariant differential operator $L^\nabla$. In $L^\nabla$, all $L$’s partial derivatives are replaced with covariant derivatives. These covariant derivatives are defined with respect to the metric tensor of the surface domain where calculations are performed. The covariant derivative of a (contravariant) vector field, $u^i(x)$, is defined as

$$u^i_x = \frac{\partial u^i}{\partial x^k} + \Gamma^i_{jk} u^k,$$

where the Christoffel symbols of the second kind are computed from derivatives of the metric tensor components $g_{jk}(x)$:

$$\Gamma^i_{jk} = (1/2)g^{il} \left( \frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^l} \right). \quad (17.24)$$

These correction terms are then used in the solution of the field equations to match one cortex with another. Note that a parameterization-invariant variational formulation could also be used to minimize metric distortion when mapping one surface to another. If $P$ and $Q$ are cortical surfaces with metric tensors $g_{jk}(u^i)$ and $h_{jk}(\xi^\alpha)$ in local coordinates $u^i$ and $\xi^\alpha$ ($i, \alpha = 1, 2$), the Dirichlet energy of the mapping $\xi(u)$ is defined as:

$$E(\xi) = \int_P e(\xi(u))dP \quad (17.25)$$

where

$$e(\xi)(u) = g^{ij}(u) \partial \xi^\alpha(u) / \partial u^i \partial \xi^\beta(u) / \partial u^j h_{\alpha\beta}(\xi(u)) \quad (17.26)$$

and $dP = (\sqrt{\det[g_{ij}]})du^1du^2$. The Euler equations, whose solution $\xi^\alpha(u)$ minimizes the mapping energy, are:

$$0 = L(\xi) = \sum_{m=1}^{2} \partial / \partial u^m [\det(g^{ij}) \sum_{i=1}^{2} g^{mi}_{\alpha\beta} \partial \xi^i / \partial u^1] \quad (17.27)$$

[143]. The resulting (harmonic) map (1) minimizes the change in metric from one surface to the other and (2) is again independent of the parameterizations (spherical